A CLASS OF AXISYMMETRIC UNSTEADY FLOWS OF VISCOUS INCOMPRESSIBLE FLUIDS

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§1. The Navier–Stokes equation in vectorial form is [1, 2]

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\Omega} \times \mathbf{v} = -\nabla H - \nu \nabla \times \boldsymbol{\Omega}, \tag{1.1}$$

where $\boldsymbol{\Omega} = \operatorname{rot} \mathbf{v}$; $H = p/\rho + v^2/2 + \Pi$; $\mathbf{v}$ is the velocity vector; t is the time; p is the pressure; ρ is the density; Π is the potential of volume forces; and ν is the kinematic viscosity coefficient.

Eddying flows of incompressible fluids with axial symmetry are considered. Having directed the z axis along the symmetry axis of the flow, a cylindrical coordinate system r, φ, z is adopted, and with the aid of the relation

$$\mathbf{v} = \nabla \times (-\mathbf{i}_\varphi \Psi/r) + \mathbf{i}_\varphi \Phi/r, \tag{1.2}$$

where $\mathbf{i}_\varphi$ is the unit vector in the direction of the azimuth angle φ, one introduces in (1.1) the stream function Ψ as well as the function $\Phi = rv_\varphi$ which characterizes the distribution of the peripheral velocity of the flow v_φ. One notices that the formula for the vector vorticity $\boldsymbol{\Omega}$ resembles in its construction the formula (1.2):

$$\boldsymbol{\Omega} = \nabla \times (\mathbf{i}_\varphi \Phi/r) + \mathbf{i}_\varphi E^2\Psi/r.$$

Equation (1.1) now splits into two parts:

$$r^{-1}\left\{\left[E^2\Psi \frac{\partial \Psi}{r\partial z} + \frac{1}{2}\frac{\partial \Phi^2}{r\partial z} - \frac{\partial}{\partial r}\left(\frac{\partial \Psi}{\partial t} - \nu E^2\Psi\right)\right]\mathbf{i}_z + \left[E^2\Psi \frac{\partial \Psi}{r\partial r} + \frac{1}{2}\frac{\partial \Phi^2}{r\partial r} - \frac{\partial}{\partial z}\left(\frac{\partial \Psi}{\partial t} - \nu E^2\Psi\right)\right]\mathbf{i}_r\right\} = \nabla H; \tag{1.3}$$

$$\partial\Phi/\partial t + r^{-1}\partial(\Psi, \Phi)/\partial(z, r) = \nu E^2\Phi. \tag{1.4}$$

In the above equations E^2 is the differential operator in the cylindrical coordinates and is given by

$$E^2 = \partial^2/\partial z^2 + \partial^2/\partial r^2 - \partial/r\partial r;$$

$\partial(\Psi, \Phi)/\partial(z, r)$ is the Jacobian of the functions Ψ and Φ.

Thus, the Navier–Stokes vector equation (1.1) is equivalent to the following system of three scalar equations for the variables Φ and Ψ:

$$\begin{cases} E^2\Psi \dfrac{\partial \Psi}{r\partial z} + \dfrac{1}{2}\dfrac{\partial \Phi^2}{r\partial z} + \dfrac{\partial}{\partial r}\left(\dfrac{\partial \Psi}{\partial t} - \nu E^2\Psi\right) = r\dfrac{\partial H}{\partial z}, \\ E^2\Psi \dfrac{\partial \Psi}{r\partial r} + \dfrac{1}{2}\dfrac{\partial \Phi^2}{r\partial r} - \dfrac{\partial}{\partial z}\left(\dfrac{\partial \Psi}{\partial t} - \nu E^2\Psi\right) = r\dfrac{\partial H}{\partial r}, \\ \dfrac{\partial \Phi}{\partial t} + r^{-1}\partial(\Psi, \Phi)/\partial(z, r) = \nu E^2\Phi. \end{cases} \tag{1.5}$$

By applying the rot operator to the left- and right-hand sides of (1.3), one finds

$$\frac{\partial}{\partial t}(E^2\Psi) + r\frac{\partial(\Psi, r^{-2}E^2\Psi)}{\partial(z, r)} - \frac{\partial \Phi^2}{r^2\partial z} = \nu E^4\Psi. \tag{1.6}$$

Thus, the kinematics of an axisymmetric flow of incompressible viscous fluid is fully described by the system of equations (1.4), (1.6), which represent a corollary of the Navier–Stokes vector equation.

These equations were introduced in [3] as a particular case of equations is curvilinear orthogonal coordinate systems.

In the case of flow with no swirl ($\Phi = 0$), Eq. (1.6) was also obtained in [2].

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A relation similar to the Bernoulli equation is derived in differential form. To this end, both sides of (1.1) are scalarly multiplied by the differential of the radius-vector $\mathbf{dR} = \mathbf{i}_z dz + \mathbf{i}_r dr + \mathbf{i}_\varphi r d\varphi$. By using (1.4), after some transformations one finds

$$dH = r^{-2}\left\{\left[E^2\Psi\frac{\partial\Psi}{\partial z} + \frac{1}{2}\frac{\partial\Phi^2}{\partial z} + r\frac{\partial}{\partial r}\left(\frac{\partial\Psi}{\partial t} - \nu E^2\Psi\right)\right]dz + \left[E^2\Psi\frac{\partial\Psi}{\partial r} - \frac{1}{2}\frac{\partial\Phi^2}{\partial r} - r\frac{\partial}{\partial z}\left(\frac{\partial\Psi}{\partial t} - \nu E^2\Psi\right)\right]dr\right\}. \tag{1.7}$$

§2. We now consider the case of axisymmetric flows of viscous fluids in which the function Φ is proportional to the stream function on Ψ, that is,

$$\Phi = k\Psi, \tag{2.1}$$

where k is a pseudoscalar constant of the dimension of the wave number.

By inserting (2.1) into (1.4) one obtains

$$k\left(\frac{\partial\Psi}{\partial t} - \nu E^2\Psi\right) = 0.$$

The expression within the brackets is defined as follows:

$$\frac{\partial\Psi}{\partial t} - \nu E^2\Psi = \begin{cases}0 & (k \neq 0),\\ -\nu A r^2 & (k = 0)\end{cases} \tag{2.2}$$

(A is a constant).

By rewriting (1.6) as

$$E^2\left(\frac{\partial\Psi}{\partial t} - \nu E^2\Psi\right) + r\frac{\partial(\Psi, r^{-2}E^2\Psi)}{\partial(z, r)} - 2k^2\Psi\frac{\partial\Psi}{r^2\partial z} = 0$$

and observing that

$$E^2 r^2 = 0, \quad -2\Psi\frac{\partial\Psi}{r^2\partial z} = r\frac{\partial(\Psi, r^{-2}\Psi)}{\partial(z, r)}.$$

one finds in view of (2.2) that

$$\frac{\partial(\Psi, r^{-2}(E^2\Psi + k^2\Psi))}{\partial(z, r)} = 0.$$

Consequently, there is a functional relation between the expression $r^{-2}(E^2\Psi + k^2\Psi)$ and the stream function Ψ in which the variables z and r do not appear explicitly.

It is advisable to set

$$E^2\Psi + k^2\Psi = r^2 F'(\Psi), \tag{2.3}$$

where $F(\Psi)$ is an arbitrary function of Ψ and the prime indicates differentiation with respect to the argument.

It follows from (1.7) that

$$F(\Psi) = \begin{cases}H - H_0(t) & (k \neq 0),\\ H + 2A\nu z - C & (k = 0).\end{cases} \tag{2.4}$$

In the above $H_0(t)$ is an arbitrary function of t and C is a constant. It is now assumed that

$$F(\Psi) = A\Psi. \tag{2.5}$$

In this case (2.2) becomes

$$\frac{\partial\Psi}{\partial t} + k^2\nu\Psi = \begin{cases}\nu A r^2 & (k \neq 0),\\ 0 & (k = 0).\end{cases} \tag{2.6}$$

It follows from the latter that for $k = 0$ (flow with no swirl) the flow is stationary and the defining stream function satisfies the equation for the stream function of an axisymmetric vortex flow of nonviscous fluid:

$$E^2\Psi = Ar^2,$$

while the modulus of the rot of a vector is proportional to the distance between the point and the symmetry axis of the flow, that is,

$$\Omega = Ar. \tag{2.7}$$

Equation (2.4) for the total energy of the (outer) mass unit of fluid for this kind of flow is

$$H = A\Psi - 2\nu Az + C.$$

It is of interest to us that in the flow under consideration the energy reserve over the entire fluid mass does not remain constant on the stream surfaces Ψ = const but depends linearly on z. Such flows were investigated in detail in [2].

By setting $\Psi = 0$ at the origin one finds that the constant C is equal to the total energy H_0 per unit mass of fluid at this point; thus, Eq. (2.7) can finally be written as

$$H = A\Psi - 2\nu Az + H_0.$$

Only the case of $k \neq 0$ and $A \neq 0$ is, in fact, new. In this case by integrating Eq. (2.6) one finds

$$\Psi = \psi(z, r) \exp(-k^2\nu t) + k^{-2}Ar^2, \tag{2.8}$$

where by virtue of (2.3) and (2.5) the function $\psi(z, r)$ satisfies the equation of a frictionless homogeneous screwlike flow with an intensity k = const and a vanishing peripheral velocity on the symmetry axis of the flow [4]:

$$E^2\psi + k^2\psi = 0. \tag{2.9}$$

The projections of the velocity and of the vector vorticity are given here by the relations

$$v_z = -\frac{\partial\psi}{r\partial r} \exp(-k^2\nu t) - 2k^{-2}A, \tag{2.10}$$

$$v_r = \frac{\partial\psi}{r\partial z} \exp(-k^2\nu t), \quad v_\varphi = \frac{k\psi}{r} \exp(-k^2\nu t) + k^{-1}Ar;$$
$$\Omega_z = -kv_z, \ \Omega_r = -kv_r, \ \Omega_\varphi = -kv_\varphi + Ar. \tag{2.11}$$

The expressions show that the components of the vectors **v** and **Ω** are collinear in an axial plane.

Thus, the flow under consideration can be represented by the superposition of two flows: a screwlike one with the stream function $\psi(z, r) \exp(-k^2\nu t)$ and the flow specified by the stream function $k^{-2}Ar^2$. The latter is analyzed below.

In the case under consideration, Eq. (2.4) assumes the form

$$H = A\Psi + H_0(t), \tag{2.12}$$

where $H_0(t)$ is the total energy per unit mass of the fluid on the flow surface $\Psi = 0$. The relation (2.12) can also be observed directly from the system (1.5).

Thus, on the flow surfaces Ψ = const the total energy per unit mass of a fluid depends only on time.

In the particular case A = 0, the relations (2.1), (2.8)-(2.12) determine a homogeneous screwlike flow of viscous fluid with vanishing azimuthal velocity on the symmetry axis. As already shown in [1], the solution of Eq. (2.9) multiplied by $\exp(-k^2\nu t)$ satisfies the Navier–Stokes equation. The conclusion that a homogeneous screwlike flow of a viscous incompressible fluid subjected to concentrated external forces can only be unsteady (damped) was confirmed in [5].

One notes that for $t \to \infty$ one has

$$\Psi = k^{-2}Ar^2, \ \Phi = k^{-1}Ar^2. \tag{2.13}$$

The above functions determine a single-parameter steady flow in which the fluid has a constant axial velocity $w = -2k^{-2}A$ and rotates as a solid body with angular velocity $\omega = {}^1/_2\Omega_z = k^{-1}A$. Flows of this type determined by functions which depend only on the distance from the point to the symmetry axis with a vanishing radial velocity component are called cylindrical [6], since circular cylinders are their Bernoulli surfaces. In the Soviet literature [4] such flows are sometimes referred to as one-parameter circulation flows or screwlike flows. Since in the case under consideration one has [6] $H = {}^1/_2w^2 + \omega^2r^2$, therefore, by comparing this equation with (2.12) one concludes that for $t \to \infty$, $H_0(t) \to {}^1/_2w^2$.

The parameters A and k can be given a specified physical sense by expressing them in terms of the axial and angular velocity of an asymptotically stable motion (2.13). The absolute value of the parameter k is equal to the ratio of the modulus of the rotation vector to the axial velocity ($k = -2\omega/w$); the value of the parameter A is given by twice the ratio of the square of the angular velocity to the axial one ($A = -2\omega^2/w$).

The first term in (2.8) can now be regarded as a function characterizing the deviation of the flow function Ψ from the flow function of a steady cylindrical flow (2.13). With time increasing this difference approaches zero due to the smoothing effect of viscosity, and the flow asymptotically approaches the steady flow (2.13) in the entire domain.

§3. A circular cylinder of radius a and of infinite length inside of which there is a swirling flow of viscous fluid moves progressively in the direction of its axis with the velocity

$$w(1 + (q/wa) \exp(-k^2 \nu t))$$

and rotates about its axis with the constant angular velocity ω. Then w is the axial velocity for an asymptotically steady motion of the cylinder when $t \to \infty$; q is a real constant; $k = -2\omega/w$.

It is obvious that in the case under consideration the flow is cylindrical and the problem consists in determining the parameters of this flow.

The motion is referred to a rigid coordinate system. The fluid flow within the cylinder can be described by using the function (2.8), where $k^{-2}A = -1/2w$ and $\psi(r)$ satisfies Eq. (2.9), that is,

$$\frac{d^2\psi}{dr^2} - \frac{d\psi}{rdr} + k^2\psi = 0. \tag{3.1}$$

The boundary conditions of the problem are

$$v_z|_{r=a} = w\left(1 + \frac{q}{wa}\exp(-k^2\nu t)\right).$$

$$v_\varphi|_{r=a} = \omega a.$$

The first condition is equivalent to

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=a} = -wa - q\exp(-k^2\nu t).$$

and the second to

$$\Psi(t, a) = k^{-1}\omega a^2.$$

One has thus arrived at the following boundary-value problem: To find a solution of Eq. (3.1) which satisfies the boundary conditions

$$\left.\frac{d\psi}{dr}\right|_{r=a} = -q; \tag{3.2}$$

$$\psi(a) = 0. \tag{3.3}$$

These two conditions must be supplemented by another one, namely, an implicit boundary condition – the velocity components have no singularities on the cylinder axis, that is, for r = 0.

These conditions are satisfied if for a particular solution of (3.1) one adopts the function

$$\psi = CrJ_1(kr), \tag{3.4}$$

where C is an arbitrary constant; J_1 is a Bessel function of the first kind and of the first order.

The condition (3.3) shows that the eigenvalues for the problem under consideration are $k_1 = \lambda_1/a$, $k_2 = \lambda_2/a, \ldots, k_n = \lambda_n/a, \ldots$, where λ_i are the zeros of the Bessel function $J_1(\lambda)$, and one need only consider positive zeros without loss of generality.

This indicates that the solution (3.4) does not describe the fluid flow in a channel for all values of w and ω, but only for those values for which the ratio $2\omega a/w$ is one of the zeros of the function $J_1(\lambda)$.

From the condition (3.2) one finds that

$$C = -q/\lambda_n J_0(\lambda_n) \quad (n = 1, 2, \ldots),$$

and, consequently,

$$\psi = -\frac{q}{\lambda_n J_0(\lambda_n)} rJ_1\left(\lambda_n \frac{r}{a}\right).$$

Therefore, the flow function for the flow under consideration in a cylindrical channel is

$$\Psi = -\frac{1}{2}wr^2 - \frac{q}{\lambda_n J_0(\lambda_n)} rJ_1\left(\lambda_n \frac{r}{a}\right)\exp\left(-\lambda_n^2 \frac{\nu t}{a^2}\right). \tag{3.5}$$

In the particular case of the progressive motion of the cylinder taking place without initial velocity one has $q = -wa$, and the expression (3.5) becomes

$$\Psi = -\frac{1}{2} wr^2\left[1 - \frac{2}{\lambda_n J_0(\lambda_n)} \frac{a}{r} J_1\left(\lambda_n \frac{r}{a}\right)\exp\left(-\lambda_n^2 \frac{vt}{a^2}\right)\right].$$

As $t \to \infty$ the effect of the wall spreads over the entire fluid which is in motion as a solid body executing a screwlike motion together with the tube.

For $w = 0$ $(\lambda_n \to \infty)$ one obtains the well-known distribution of circular velocity on the steady motion within the uniformly moving circular cylinder $v_\varphi = \omega r$.

A complete lack of motion of the cylinder as well as of the fluid corresponds to the relation $\omega = 0$ $(\lambda_0 = 0)$.

§4. The stream function ψ_- obtained in [7] for the motion of a frictionless fluid within a spherical screwlike vortex is multiplied by $\exp(-k^2\nu t)$. Then, by virtue of the Steklov theorem mentioned in Sec. 2, one finds the stream function Ψ_-, which describes the screwlike motion of a viscous fluid inside a sphere. By "gluing" together this solution and the outer potential flow, one comes to the conclusion that the incoming flow on the static vortex must have the velocity $w_0 \exp(-k^2\nu t)$ and

$$\Psi_- = \frac{3}{2} w_0 \exp(-k^2 \nu t) \frac{a^{3/2}}{bJ_{1/2}(b)} R^{1/2} J_{3/2}\left(b\frac{R}{a}\right)\sin^2\theta \quad (R < a),$$

where w_0 is the initial velocity of the accumulated flow; $J_{1/2}$ and $J_{3/2}$ are the Bessel functions of the orders $1/2$ and $3/2$; R, θ are the spherical coordinates of a point; a is the vortex radius; $b = 4.4934$ is the lowest positive root of the function $J_{3/2}(\lambda)$; $k = b/a$.

The velocity components of the inner motion are also obtained from the respective velocity components of the frictionless motion [7] by multiplying the latter by $\exp(-b^2\nu t/a^2)$.

The stream function outside the vortex is

$$\Psi_+ = \frac{1}{2} w_0 \exp(-b^2 \nu t/a^2)(1 - a^3/R^3) R^2 \sin^2\theta \quad (R > a).$$

The multiplier $\exp(-b^2\nu t/a^2)$ reflects the effect of the viscous flow inside the vortex on the frictionless flow outside it.

Since the fluid outside the vortex is frictionless and the motion is unsteady and rotationless, therefore, the pressure on the rotation boundary from the outside can be found with the aid of the Lagrange–Cauchy integral:

$$H_+ = f(t) - \frac{3}{2} b^2 \nu a^{-1} w_0 \exp\left(-b^2\frac{\nu t}{a^2}\right)\cos\theta,$$

where $f(t)$ is an arbitrary time function.

At the same time, the pressure on the boundary from within the rotation is determined with the aid of Eq. (2.12), which in this case assumes a particular simple form:

$$H_- = H_0(t).$$

If it is required that the total energy of the fluid mass on the sphere surface be continuous, that is, that the condition

$$H_+ + \partial\varphi/\partial t = H_- \text{ for } \quad R = a$$

be satisfied, one finds that

$$H_0(t) = f(t) = H_+^*(t),$$

where H_+^* is the value of the trinomial H_+ in the equatorial plane $(\theta = \pi/2)$ of the vortex.

Since the velocity is continuous, the pressure on the boundary must suffer a discontinuity (excluding the points of the equator). A similar effect occurs in the case of a Hill spherical vortex which carries the viscous fluid in the frictionless medium [2].

A jump in the normal stress results in a rotational strain which is ignored here. In the case of a droplet it is compensated for by the pressure surface tension, which is expected to be sufficiently strong to maintain the spherical shape of a droplet.

Reversing the motion analyzed at this point, one obtains a spherical screwlike vortex filled by a viscous fluid and moving in a straight line in a frictionless fluid, steady at infinity, with the velocity $w_0 \exp \times (-b^2\nu t/a^2)$ in the direction of the z axis.

The reaction of the fluid to this vortex in the projection onto the direction of its motion is

$$R_z = \frac{2}{3} b^2 \pi a \mu w_0 \exp(-b^2 \nu t/a^2).$$

This force decreases in the course of time the more rapidly, the smaller the vortex radius and the greater the viscosity of the fluid filling it.

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DIFFUSION OF POLYMER SOLUTIONS IN A TURBULENT BOUNDARY LAYER

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In the last 10 years much progress has been made with the experimental investigation of the Toms effect – the reduced friction in turbulent flows containing small amounts of high-molecular-weight compounds (polymers). However, most experiments have been made at a constant polymer concentration, e.g., in connection with the flow of previously prepared solutions through pipes and channels. Much less attention has been paid to the more complicated but very important practical situation in which a polymer is introduced into a turbulent boundary layer (TBL) through a surface slit. In this case, as a result of turbulent diffusion, the polymer concentration falls off both downstream from the slit and in a direction normal to the surface. On the one hand, the diffusion of the polymer depends on the turbulent mixing capability of the flow, while, on the other hand, it directly affects that capability. This explains why the diffusion of active admixtures in turbulent flows is more complicated and has been less studied than that of passive admixtures that do not affect the flow.

Qualitatively, for both active and passive admixtures, the diffusion process in TBL is characterized by the existence of three zones along the flow. In the initial zone, nearest to the slit source, the admixture diffuses from the wall to the outer edge of the viscous sublayer. In the following intermediate zone the admixture progressively occupies the whole of the TBL and the thickness of the diffusion layer approaches the thickness of the dynamic layer. This stage is followed by diffusion in the end zone.

For both active and passive admixtures the initial zone is very short and at $q \geq 50\nu$ it is totally absent (q is the solution flow rate per unit length of the slit and ν is the kinematic viscosity of the flow, so that the right side of the inequality is the flow rate in the viscous sublayer). For passive admixtures the intermediate zone is also small (about 60-80 TBL thicknesses) [1]. Accordingly, in the diffusion calculations for passive admixtures it is usual to ignore the presence of the first two zones and take into account only the third, most

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